



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

***The Representation of Finite Groups, especially of the
Rotation Groups of the Regular Bodies of Three-
and Four-dimensional Space, by
Cayley's Color Diagrams.***

BY H. MASCHKE.

The graphical representation of a group given by Cayley* leads to a diagram consisting of several lines of different colors, a so-called *color-group*, which affords a very clear insight into the structure of the group. Cayley himself applied his method only to groups of comparatively low orders, and it seems that the method has never been used for more complicated cases.†

The purpose of the present paper is to show how readily Cayley's method can be applied to the construction and investigation of numerous groups of higher orders. In particular, the color diagrams for the rotation groups of the regular bodies can be arranged in such a way that they lend themselves much easier, at least in some respects, to a study of the groups concerned, than even the models of the regular bodies.

The most prominent feature of these diagrams, to which their high degree of perspicuity is due, consists in the fact that their color lines do not intersect each other, so that the diagrams, when described on the sphere, constitute convex polyhedrons. I determine, in the first part of the paper, all the color-groups thus defined and show that, apart from two other cases, they are identical with the rotation groups of the regular bodies. In the second part I study in detail the connection between the rotation groups and the corresponding diagrams. The third part of the paper contains some extensions of the

* Cayley, "The Theory of Groups: graphical representation," American Journal, vol. I, p. 174; "On the Theory of Groups," American Journal, vol. XI, p. 139.

† Cf. Young, "On the Determination of Groups whose Order is a Power of a Prime," American Journal, vol. XV, p. 48, where the color diagrams for the groups of order 16 are given.

principles developed, leading to the so-called extended rotation groups and other groups of a similar character. In the last part I determine the color diagrams for the rotation groups of the regular four-dimensional bodies, which I define analytically by orthogonal substitutions, showing also the peculiar connection between these groups and the theory of the Icosahedron.

It seems desirable to give at the outset a short explanation of Cayley's method, modified in as much as it bears on the present subject.

PART I.

§1.—*Cayley's Method.*

Let a finite group G of order N be generated by all the possible combinations and repetitions of a certain number of "generating" operations S, T, \dots, U . These generators are supposed to be independent of each other, i. e. it must not be possible to express any of them in terms of the others. Now let O_1 be any object capable of being acted upon by the operations of G , and such that none of the operations of G , except unity, leaves O_1 unchanged. Represent this object by a point a_1 . When S is applied, O_1 may be transformed into O_2 , and O_2 represented by another point a_2 . Applying then in succession all the different operations of G , we obtain, corresponding to O_1, O_2, \dots, O_N , a system of N distinct points a_1, a_2, \dots, a_N , which we will call the *fundamental points*. The operation S changes every point a_k into another point. Otherwise, denoting by V the operation transforming a_1 into a_k , $VS.V^{-1}$ would leave the point a_1 , i. e. the object O_1 unchanged. But since O_1 , according to our hypothesis, remains unchanged only by the operation unity, we would have $VSV^{-1} = 1$, or $S = 1$. We may, therefore, indicate the operation S by directed lines (lines with an arrow) leading from every point to that into which it is transformed by S . But S has a finite period, say p , so that $S^p (= 1)$ leaves every point unaltered. The system of lines representing the operation S consists accordingly of N/p polygons of p sides. Similarly, if q is the period of T , we obtain a system of lines representing the operation T consisting of N/q polygons of q sides, etc. To distinguish the S -lines from the T -lines, etc., we choose different colors for the lines belonging to different generating operations. Such a system of directed lines of different colors, representing a group, has been

called by Cayley a *color-group*.* It has the following important properties which will be referred to as theorems 1 and 2:

Theorem 1. *To every point and from every point there leads one and only one line of each color.*

Every operation $S^a T^b U^c \dots$ determines therefore uniquely a *route* to be described from any initial point a . The end point of this route indicates the effect of the operation on the point a .

Theorem 2. *Every route (defined by $S^a T^b U^c \dots = 1$) leading from any point a back to the same point a , leads from every point a_k back to the same point a_k .*

The conditions given by these two theorems are not only necessary, but also sufficient for a diagram to constitute a color-group.

If the period of one of the generating operations, say T , be 2, then the $N/2$ polygons representing T are bilateral. Every one of these "digons" may be indicated by *one* (double) line of a certain color, having two directions, and we shall agree in this case to simply omit the arrows. The above two theorems hold, of course, also in this case.

§2.—*Regular Color-groups.*

I now propose to find all those *two-colored* color-groups in the plane, *the lines of which do not intersect each other*, except, of course, in the fundamental points. Such a color-group I shall hereafter call a *regular* color-group. The plane may be transformed into a sphere (we are in this whole investigation only concerned with geometry of *analysis situs*) and then we obtain a convex polyhedron on the sphere. Conversely, to any two-colored color-group on the sphere, forming a convex polyhedron, there corresponds a regular color-group in the plane.

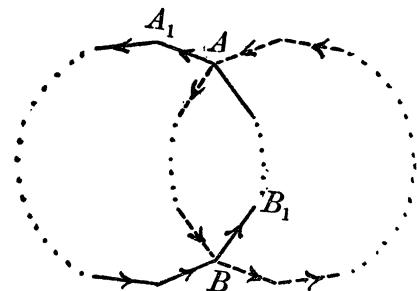
The two generating operations with which we are now only concerned may be denoted by S -black lines and R -red lines. In the adjoined figures a black line, or an S -line, as we sometimes shall call it, will always be represented by a continuous line, ———, a red or R -line by a broken line, -----.

With regard to these regular color-groups, I am going to prove two fundamental theorems which may be referred to as theorems 3 and 4.

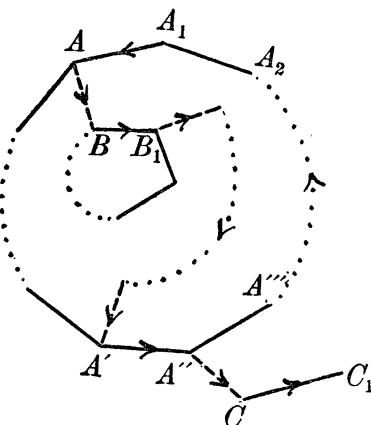
*This word will be used in the sense of the geometric diagram as well as in the sense of the abstract group represented by the diagram.

Theorem 3. The four-colored lines concurring in one point are such that the two lines of one color are not separated by a line of the other color.

If it were otherwise, then we would have a black and red polygon meeting in two points A and B , as shown in the adjoined figure.



A black and a red route would then lead in the direction of the arrows from A to B , involving an equation $S^a = R^b$, or $S^a R^{-b} = 1$. Apply now the route $S^a R^{-b}$ to A_1 . The route S^a leads to B_1 . But B_1 cannot be identical with the point A , otherwise we would have a relation $S = R^a$. Now R^{-b} must, on account of $S^a R^{-b} = 1$, lead from B_1 back to A_1 , which, on the other hand, is impossible, since one of the points A_1 and B_1 lies outside, the other inside the red polygon, and R^{-b} is a red route which cannot cross the circumference of the red polygon. If it would, then four red lines would concur in one fundamental point, in contradiction to theorem 1.



Theorem 4. Any S - or R -polygon contains in its interior either all the points of the color-group or no points at all.

Let $AA_1 \dots A''A' \dots$ be a black polygon P , and suppose some points of

the color-group lie within P . At least one red line must lead from some corner of P into the interior of P , otherwise the points within P would not be connected at all with the corners of P , whereas it must be possible to pass from every point of the color-group to every other. Let this red line be the line AB . The point B must belong to some S -polygon which lies entirely inside P .

Let us now consider the operation RS . This operation has a finite period, say λ , so that $(RS)^\lambda = 1$. If we apply the route $(RS)^\lambda$ to the point A , then this route must lead back to A , but it may happen to lie partly outside P . In this case the route leading from A to B , then to B_1 , etc., must meet some point, say A' of P , then lead to A'' , C , C_1 , and finally from C_1 back to A_1 and A . That part of this route which leads from A to A'' is given by $(RS)^{\lambda-\mu}$, and now we may go back from A'' to A directly on the black polygon P , i. e. by the route S^ν . Thus we have $(RS)^{\lambda-\mu}S^\nu = 1$. If now we apply $(RS)^{\lambda-\mu}S^\nu$ to B_1 , then $(RS)^{\lambda-\mu}$ leads from B_1 to C_1 , and the black route S^ν ought to lead back from C_1 to B_1 , which, however, is impossible, C_1 lying outside, B_1 inside the black polygon P .

The consequence is that the route $(RS)^\lambda$ applied to A lies entirely inside P . It leads accordingly back by a red line to A_1 and then to A . Thus the existence of a red line leading from A into the interior of P involves the existence of a red line leading from the interior of P to A_1 . But according to theorem 3 the other red line concurring in A_1 must also lie inside P . We have then also in A_1 a red line leading into the interior of P . That implies again the existence of two red lines emanating from A_2 into the interior of P and so on, all around the polygon P . The result is that all the red lines concurring in all the corners of P lie inside P . There is no connection left for any outside points, and therefore the polygon P contains all the fundamental points of the color-group in its interior. Thus theorem 4 is proved.

If we lay out the color diagram on the sphere, then we obtain a convex polyhedron containing a certain number of S - and R -polygons which all exclude each other on account of theorems 3 and 4. For that polygon which includes all the other points of the color-group in the plane appears now on the sphere as a polygon, including, on its other side, no point.

Besides the S - and R -polygons, there are other polygons on the color polyhedron. The sides of every one of those are alternately black and red (and therefore of even number) in consequence of theorems 3 and 4, as an easy consideration shows. These polygons we shall call *intermediate*.

For further investigation we apply Euler's theorem. Two cases are to be distinguished—

- 1). S and R are both of a period higher than 2.
- 2). S or R or both are of period 2.

§3.—*The Periods of R and S are higher than 2.*

In this case we have in every vertex of the color polyhedron four concurring edges. The number of vertices is N , the order of the group. The number of edges is $4N/2 = 2N$. The number of faces, f , is, according to Euler's theorem, $f = 2N - N + 2 = N + 2$.

Denote the number of trilateral, quadrilateral, etc., faces by f_3, f_4, \dots , then we have

$$f_3 + f_4 + f_5 + \dots = N + 2, \quad (1)$$

$$3f_3 + 4f_4 + 5f_5 + \dots = 4N. \quad (2)$$

Multiplying the first equation by 4 and subtracting the second, we obtain

$$f_3 = 8 + (f_5 + 2f_6 + \dots).$$

Consequently some of our polygons must be triangles. But all the intermediate polygons have an even number of sides. Therefore either the S -polygons or the R -polygons or both must be triangles, i. e.

The period of at least one of the two generating operations S and R is 3.

If, now, the S - and R -polygons be triangles, then

$$f_3 = 2N/3, f_5 = f_7 = f_9 = \dots = 0,$$

so that

$$3f_4 + 3f_6 + 3f_8 + \dots = N + 6,$$

$$2f_4 + 3f_6 + 4f_8 + \dots = N,$$

whence we deduce

$$f_4 = 6 + (f_8 + 2f_{10} + \dots),$$

i. e. *some of the intermediate polygons are quadrangles.*

If, on the other hand, only the S -polygons be triangles, then we have $f_3 = N/3$ and

$$f_4 + f_5 + f_6 + \dots = 2N/3 + 2,$$

$$4f_4 + 5f_5 + 6f_6 + \dots = 3N.$$

Multiplying the first of these equations by 5 and subtracting the second, we obtain

$$f_4 = N/3 + 10 + (f_6 + 2f_7 + \dots), \text{ i. e. } f_4 > N/3.$$

Now the R -polygons might be quadrangles, but if so, then their number is only $N/4$. Consequently also in this case, *some of the intermediate polygons must be quadrangles.*

This result means—

- 1) there exists a relation $R^{\epsilon_1}S^{\epsilon_2}R^{\epsilon_3}S^{\epsilon_4} = 1$, where the quantities ϵ stand for $+1$ or -1 ;
- 2) there exist some quadrilaterals, defined by the closed route $R^{\epsilon_1}S^{\epsilon_2}R^{\epsilon_3}S^{\epsilon_4}$, which are quadrangles, i. e. which do not include any other points of the color-group.*

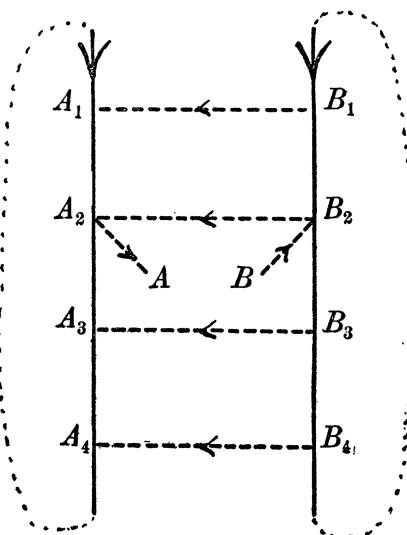
It will be shown that all the intermediate polygons are quadrangles.

We may assume $\epsilon_1 = \epsilon_2 = +1$, for if U denotes one of the generating operations of a group, it may be replaced by U^{-1} , so that in the case ϵ_1 or $\epsilon_2 = -1$ we have only to change the signification of R and S into R^{-1} and S^{-1} .

Examining first the case

$$RSR^{-1}S^{-1} = 1, \quad (3)$$

let $B_1A_1A_2B_2$ be a quadrangle corresponding to relation (3). B_1B_2 and A_1A_2 belong each to a black polygon $B_1B_2B_3 \dots$ and $A_1A_2A_3 \dots$ resp., as shown in the figure, where the direction in which these polygons are to be described is indicated by only one arrow. Applying now the route (3) to the points $B_2, B_3, B_4 \dots$ in succession, we see that all these points are connected with the corresponding points $A_2, A_3, A_4 \dots$ by equally directed red lines. Besides



* I distinguish between a quadrilateral, i. e. any closed route consisting of four lines, and a quadrangle, i. e. a quadrilateral face of the polyhedron.

$A_2 B_2$, another red line $A_2 A$ will concur in A_2 , and this line cannot lead into the interior of the black polygon $A_1 A_2 A_3 \dots$ according to theorem 3, nor into the interior of $A_1 A_2 B_2 B_1$, which is, according to our hypothesis, a quadrangle. It leads, therefore, into the interior of the quadrilateral $A_2 A_3 B_3 B_2$. The same argument holds with regard to the red line $B_2 B$. From (3) we derive at once $R^{-1}S^{-1}RS = 1$, a route which is represented, for instance, by $A_2 B_2 B_1 A_1 A_2$. But if we try to apply this route to A , then $R^{-1}S^{-1}$ leads to A_1 , and now there remains no possibility to return from A_1 by RS to A . Relation (3) is therefore to be rejected.

By a similar argumentation it follows that also the relation

$$RSR^{-1}S = 1 \quad (4)$$

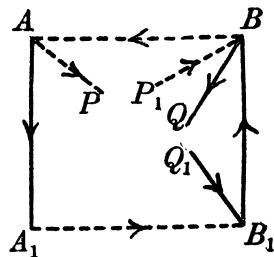
is to be rejected. To show this, we have in the figure only to reverse the arrow of the polygon $B_1 B_2 B_3 \dots$ and to apply the route (4) to the point B . RS leads to B_1 , and we cannot return from B_1 by $R^{-1}S$ to B .

Interchanging in (4) R and S , we see that also the case $SRS^{-1}R = 1$, or $RSRS^{-1} = 1$, is to be rejected.

There remains then only

$$RSRS = 1. \quad (5)$$

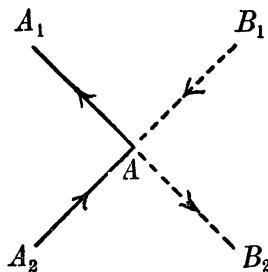
Now if any quadrilateral BAA_1B_1 (see the following figure) defined by (5) contain any inside point, then at least one red or black line must lead from some corner to the interior, for instance, the red line AP . It follows that also P_1 , defined by $P_1B \equiv R$, must be an inside point, because it belongs with PAB to the same red polygon. Applying now (5) to P_1 , we see that RS cannot lead to any outside point, for then, in order to return from this outside point by the route RS to P_1 , the route R could only lead to A_1 , and S from A_1 directly to P_1 ;



but this would give rise to the closed route $P_1BB_1A_1$, i. e. to $RS^{-1}R^{-1}S = 1$, which is impossible. Accordingly a black line BQ must lead to the interior,

and likewise $Q_1 B_1$ from the interior to B_1 . Finally we see, applying the same argument to Q_1 , etc., that all the black and red lines emanating from the four points $AB_1 A_1$ lead to the inside.

If, therefore, any quadrilateral defined by (5) contain any inside points, then no point of the color-group will be outside this quadrilateral, and this means, considering the color-group as a polyhedron on the sphere, no quadrilateral $RSRS$ contains any points in its interior, or every such quadrilateral is an intermediate quadrangle.



Now if we consider any vertex A with the four concurring edges AA_1 , AA_2 , AB_1 , AB_2 , then $B_1 A A_1$ are three points of an intermediate quadrangle, and such are also $A_2 A B_2$, corresponding to the route (5). The same holds for every fundamental point of the color-group, and thus we see that *all the intermediate polygons are quadrangles.*

According to the preceding results either the S - or the R -polygons, say the S -polygons, are triangles. Their number f_3 is given by the equation $3f_3 = N$. The number of the intermediate quadrangles f_4 is given by the equation $2f_4 = N$, because every vertex belongs to two quadrangles. Let the R -polygons have x sides, and let f_x be their number. Then we have $xf_x = N$. Equation (1) becomes now

$$f_3 + f_4 + f_x = N + 2,$$

i. e. $\frac{N}{3} + \frac{N}{2} + \frac{N}{x} = N + 2,$

or $N(6 - x) = 12x.$

But $x \geq 3$, according to our hypothesis, and since $6 - x$ in the last equation must be positive, x can assume only the values 3, 4, 5. Thus we obtain the following cases:

- 1) $x = 3, N = 12,$
- 2) $x = 4, N = 24,$
- 3) $x = 5, N = 60.$

The corresponding color-groups are now defined completely; they are given in Figs. 4, 7 and 10, and will be denoted resp. by V, VII and X. We observe, viewing the color polyhedron from points outside the sphere, that in all the S -polygons the arrows run the same way and also in all the R -polygons. This fact is due to the relation $RSRS=1$ or $RS^{-1}RS^{-1}=1$. We postpone a further explanation of these color-groups to §5, where they will be treated together with the results of §4.

§4.—*The Period of R or S is equal to 2.*

If the generators R and S are both of period 2, then we have $N=2n$ (n any integer). The corresponding color-group is given by a closed polygon of $2n$ sides, consisting alternately of black and red lines—see Fig. 1; this color-group will be denoted by I. Either line is a not directed double line which may be described either way.

We now assume only one generator, say R of period 2, the other S of period n ($n > 2$). Suppose, at first, any two S -polygons, $A_1 A_2 A_3 \dots$ and $B_1 B_2 B_3 \dots$ be connected by two red lines, $A_1 B_1$ and $A_k B_k$. Then some relation $S^a RS^b R = 1$ holds, representing the closed route $A_1 A_2 B_k B_1 A_1$. If we apply the same route to A_2 , we see that A_2 must be connected by a red line with some point B of the other polygon, and further, that *each* point A of the first polygon must be connected with some point of the second polygon. But the same reasoning applies to the second polygon. Thus we see *if two S -polygons hang together by more than one red line, then all the points of the two polygons are connected by red lines.*

But then all the points of the two polygons are, so to speak, “saturated”; they form a color-group consisting of $N=2n$ points, where the intermediate polygons are quadrangles, satisfying either the relation $SRSR=1$ or $SRS^{-1}R=1$. The color-group for the first case ($SRSR=1$) is given in Fig. 2 for $n=3$, and will be denoted by II. The diagram for the case $SRS^{-1}R=1$ differs from Fig. 2 only by the reversed direction of the sides of the inside or outside S -polygon. We will denote this color-group by IIa.

In the following we suppose, then, that any two S -polygons are connected at the utmost by *one* red line, and accordingly, that the lowest number of sides of the intermediate polygons be 6.

The number of edges is $3N/2$, since there are in every vertex 3 concurring edges. Euler's theorem gives the number of faces $f = N/2 + 2$. Thus we obtain these two equations

$$f_3 + f_4 + f_5 + f_6 + \dots = N/2 + 2, \quad (6)$$

$$3f_3 + 4f_4 + 5f_5 + 6f_6 + \dots = 3N. \quad (7)$$

Multiplying equation (6) by 6 and subtracting (7), we have

$$3f_3 + 2f_4 + f_5 = 12 + (f_7 + 2f_8 + \dots),$$

and this equation shows the existence of some S -polygons of either 5 or 4 or 3 sides, since the number of sides of the intermediate polygons is even, and the existence of quadrangles as intermediate polygons has already been excluded.

The period of S is then either 5 or 4 or 3.

$$1) \quad S^5 = 1.$$

We have in this case $f_5 = N/5$, $f_3 = 0$, $f_4 = 0$, and equations (6) and (7) become

$$f_6 + f_8 + f_{10} + \dots = 3N/10 + 2,$$

$$6f_6 + 8f_8 + 10f_{10} + \dots = 2N,$$

whence, multiplying the first by 8 and subtracting the second,

$$2f_6 = 2N/5 + 2 + (2f_{10} + 4f_{12} + \dots).$$

There must, consequently, exist *some intermediate polygons of 6 sides*.

$$2) \quad S^4 = 1.$$

We have here $f_4 = N/4$, $f_3 = 0$, $f_5 = 0$,

$$f_6 + f_8 + f_{10} + \dots = N/4 + 2,$$

$$3f_6 + 4f_8 + 5f_{10} + \dots = N,$$

$$\therefore f_6 = 8 + (f_{10} + \dots).$$

Also in this case there must exist *some intermediate hexagons*.

$$3) \quad S^3 = 1.$$

We have $f_3 = N/3$, $f_4 = 0$, $f_5 = 0$,

$$f_6 + f_8 + f_{10} + \dots = N/6 + 2,$$

$$3f_6 + 4f_8 + 5f_{10} + \dots = N,$$

$$\therefore 3f_6 + 2f_8 + f_{10} = 12 + (f_{12} + \dots).$$

Here we can only infer that *there must exist intermediate polygons of*

a) 6, or b) 8, or c) 10 sides.

In the cases 1), 2) and 3a) there exist some intermediate hexagons. This involves one of the two relations

$$SRSRS^{-1}R = 1, \quad (8)$$

$$SRSRSP = 1, \quad (9)$$

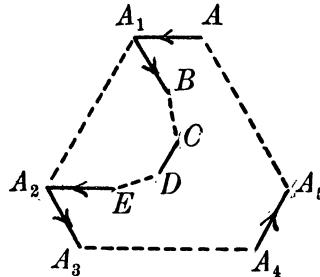
the relation $SRS^{-1}RS^{-1}R = 1$ being essentially the same as (8) and $S^{-1}RS^{-1}RS^{-1}R = 1$ the same as (9). From (8) we deduce

$$RSRS^{-1}RS = 1,$$

and combining this relation with (8), we obtain

$$SRSRS^{-1}R \cdot RSRS^{-1}RS = 1,$$

which reduces to $S^2 = 1$. But the period of S is, according to our hypothesis, > 2 , and therefore case (8) is to be rejected. There remains relation (9).



Let now $AA_1A_2 \dots A_5A$, as shown in the adjoined figure, be a route given by $(SR)^3 = 1$. If there were any points inside this polygon P , then some black line, say A_1B , would lead into the interior of P . Apply now $(SR)^3$ to point A_1 . S leads to B , R from B to a point C , again inside P , S from C to a point D , which cannot be any one of the corner points of P , for then there would arise two triangles connected by two red lines. R leads from D again to an inside point E and finally S to A_2 , because R must lead back to A_1 . Consequently the S -polygon belonging to A_2A_3 lies inside the polygon P , and, by a similar reasoning, also that which belongs to A_4A_5 .

There is then no connection whatever with any outside point, and considering again the color-group on the sphere, we see that all the polygons $(SR)^3 = 1$ must be intermediate polygons. It follows that conversely every intermediate

polygon is given by the relation $(SR)^3 = 1$. The conclusion is entirely analogous to that connected with the figure on page 163.

Now we have in case 1, page 166, f_5 pentagons and f_6 hexagons. The number of vertices being N , we obtain $5f_5 = N$ and $6f_6/2 = N$, because each vertex lies on 2 adjoining hexagons. Euler's theorem gives

$$\begin{aligned} f_5 + f_6 &= N/2 + 2, \\ \text{or} \quad \frac{N}{4} + \frac{N}{3} &= \frac{N}{2} + 2, \\ \therefore N = 60, f_5 = 12, f_6 &= 20. \end{aligned}$$

This defines completely the color-group VIII given in Fig. 8.

Case 2, page 166, gives $4f_4 = N$, $6f_6/2 = N$,

$$\begin{aligned} f_4 + f_6 &= N/2 + 2, \\ \frac{N}{4} + \frac{N}{3} &= \frac{N}{2} + 2, \\ \therefore N = 24, f_4 = 6, f_6 &= 8. \end{aligned}$$

The corresponding color-group V is given in Fig. 5.

In case 3a, page 166, we have $3f_3 = N$, $6f_6/2 = N$,

$$\begin{aligned} f_3 + f_6 &= N/2 + 2, \\ \frac{N}{3} + \frac{N}{3} &= \frac{N}{2} + 2, \\ \therefore N = 12, f_3 = 4, f_6 &= 4, \end{aligned}$$

Fig. 3 represents this color-group III.

There remain the two cases 3b and 3c, page 166. In case 3b there exist some intermediate octagons. This involves one of the following relations:

$$SRSRSRSR = 1, \tag{10}$$

$$SRS^{-1}RSRS^{-1}R = 1, \tag{11}$$

$$SRSRS^{-1}RS^{-1}R = 1, \tag{12}$$

$$SRSRSR S^{-1}R = 1. \tag{13}$$

Relation (13) is to be rejected, for if we combine it with $RSRS^{-1}RSRS = 1$, which is an immediate consequence of it, we obtain

$$SRSRSRS^{-1}R \cdot RSRS^{-1}RSRS = 1,$$

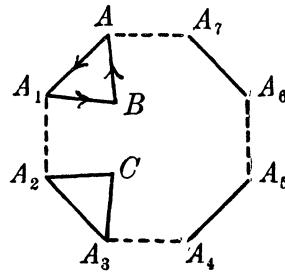
which reduces to

$$SRS^2RS = 1,$$

or, the period of S being 3,

$$RS^2RS^2 = 1.$$

But this relation would give rise to a connection of two black triangles by two red lines.



In the three remaining cases (10), (11), (12), we suppose that an octagon $AA_1A_2A_3 \dots A_7A$, defined by one of these equations, contains inside points. Then at least one of the four S -triangles, say AA_1B , must lie in the interior of the octagon P . Apply now

$$SRS^{\alpha_1}RS^{\alpha_2}RS^{\alpha_3}R = 1 \quad (14)$$

—where $\alpha_1, \alpha_2, \alpha_3 = \pm 1$ according to (10), (11) or (12)—to the point A_1 . This route must lie entirely inside the octagon, because otherwise there would arise a hexagon or a quadrangle involving some relation $S^{\alpha_1}RS^{\alpha_2}RS^{\alpha_3}R = 1$ or $S^{\alpha_1}RS^{\alpha_2}R = 1$. The route (14) leads, then, from A_1 into the interior of the octagon, then to A_2 and back to A_1 . In case (10) and (11) we see at once that the route must lead to A_2 from an inside point C (not from A_3). In case (12) we apply the route $SRS^{-1}RS^{-1}RSR = 1$ to A_1 and come to the same conclusion. The triangle belonging to A_2A_3 lies therefore in all these cases inside the octagon, and similarly also the triangles belonging to A_4A_5 and A_6A_7 .

Consequently all the polygons defined by the equations (10), (11) or (12) are intermediate polygons, and all the intermediate polygons are given by one of these equations.

We have now $3f_3 = N$, $8f_8/2 = N$,

$$\begin{aligned} f_3 + f_8 &= N/2 + 2, \\ \frac{N}{3} + \frac{N}{4} &= \frac{N}{2} + 2, \\ \therefore N &= 24, f_3 = 8, f_8 = 6. \end{aligned}$$

The corresponding color-group VI is given in Fig. 6. The arrows in that

diagram are taken according to the relation (10). Taking the directions of the arrows of the eight triangles alternately opposite, we obtain the color-group belonging to (11), which may be denoted by VIa. Case (12), however, is to be rejected altogether, since it is impossible to arrange in each intermediate octagon the arrows according to (12).

In the remaining case 3c, page 166, we have the following possibilities :

$$SRSRSRSRSR = 1, \quad (15)$$

$$SRSRSRSR S^{-1}R = 1, \quad (16)$$

$$SRSRSRS^{-1}RS^{-1}R = 1, \quad (17)$$

$$SRSRS^{-1}RSRS^{-1}R = 1. \quad (18)$$

In case (16) we have $SRSRSR = RSRS^{-1}$ and $SRSRSRS^{-1}RSR = 1$. Combining these two relations, there follows

$$RSRS^{-1}S^{-1}RSR = 1,$$

or $RS^{-2}RS = 1,$

or $RS^{-2}RS^2 = 1,$

which relation would lead to two triangles connected by two red lines. Case (16) is therefore to be rejected.

In case (17) there follows $RSRSRS^{-1}RS^{-1}RS = 1$, and multiplying this on the left by (17), the product reduces to $S^2 = 1$, which shows that also case (17) is to be rejected.

Case (18) leads to the same contradiction $S^2 = 1$, as can be seen by multiplying equation (18) on the right-hand by $RSRS^{-1}RSRS^{-1}RS = 1$.

There remains therefore only relation (15). That any decagon corresponding to this relation cannot contain any inside points can be shown in exactly the same way as the corresponding proposition has been proved in the case of the octagons defined by (10). All the intermediate polygons are then defined by (15), and we have $3f_3 = N$, $10f_{10}/2 = N$,

$$\begin{aligned} f_3 + f_{10} &= N/2 + 2, \\ \frac{N}{3} + \frac{N}{5} &= \frac{N}{2} + 2, \\ \therefore N &= 60, \quad f_3 = 20, \quad f_{10} = 12. \end{aligned}$$

The corresponding color-group IX is given in Fig. 9.

Reviewing the color-groups obtained in this section (§4), we observe that in every one of the color-groups II, III, V, VI, VIII and IX, represented by Figs. 2, 3, 5, 6, 8 and 9 resp., the S -polygons are equally directed, while in the groups IIa and VIa, the diagrams of which, except the arrows, are also given in Figs. 2 and 6 resp., the directions of the S -polygons are alternately opposite.

PART II.

§5.—*Connection with the Rotation Groups of the Regular Bodies.*

The result of the preceding developments can be stated as follows:

All those two-colored color-groups of a finite number of fundamental points whose diagrams constitute a convex polyhedron on the sphere are represented by the groups I, II, IIa, III, IV, V, VI, VIa, VII, VIII, IX, X, given in Figs. 1–10 in plane projection. (Regarding the diagrams of IIa and VIa see the remark at the end of §4.)*

There is an obvious connection between these diagrams laid out on the sphere and the figures of the regular bodies, viz.

The diagrams 3, 5, 6, 8 and 9 represent a tetrahedron, octahedron, hexahedron, icosahedron and dodecahedron with truncated vertices; the diagrams 4, 7 and 10 a tetrahedron, octahedron and icosahedron whose edges are truncated and the vertices so far as to yield polygons of so many sides as there were edges concurring in the original vertices.

The above given result could have been deduced without any difficulty, if it were admissible to suppose the so-called “intermediate” polygons in each color-group to be of equal number of sides. The demonstration of this important property of the intermediate polygons may be regarded as the principal object of the preceding investigation.

This close connection of our diagrams with the regular bodies suggests the existence of some connection between our color-groups and the rotation groups of the regular bodies. I am going to show that the color-groups I–X represent precisely these rotation groups. The color-groups IIa and VIa, being of a different character, will be treated separately in §6.

* The color-groups of the type I, II and IIa have also been given, in a slightly different form, by Cayley, *American Journal*, vol. XI, p. 154–155.

We mark on the faces of the regular bodies a *general point group*.* This means we choose arbitrarily, say in the vicinity of one of the vertices, a point, and let the regular body undergo all the N rotations of the corresponding group. By this process the originally selected point assumes altogether N different positions determining N points, the points of a "general point group." *These N points we take as the fundamental points of the corresponding color-group.*

With regard to notation we denote the faces by letters, the vertices by numbers, and accordingly, for instance, the point situated on the face a in the vicinity of the vertex 1 by a_1 , etc.

The Dihedron (cf. Figs. 1, 2 and 11).

In Fig. 11 the upper and lower half of the dihedron for the case $n=3$ is shown separately.

The color-group I, Fig. 1, represents the dihedron group generated by two rotations of period 2. The black lines signify a rotation U of period 2 about the diameter bisecting the edge which is common to the two faces b , the red lines a rotation T of period 2 about the diameter bisecting the edge which is common to the two faces a .

We see from the color-group in Fig. 1 as well as from the model in Fig. 11 that T transforms the point a_1 into a_2 , b_1 into c_2 , c_1 into b_2 , U the point a_1 into c_2 , b_1 into b_2 , c_1 into a_2 , etc.

In the color-group II, Fig. 2, the black lines represent a rotation S of period 3 about the principal dihedron axis 1, 2; the red lines the same rotation T as in I.

Comparing the color-groups I and II, we deduce at once the relation $UT=S$; we see also from Fig. 1 immediately that UT is of period 3, from Fig. 2 that ST is of period 2, etc.

The Tetrahedron (cf. Figs. 3, 4 and 12).

The left part of Fig. 12 represents the upper pyramid of the tetrahedron; the right part, the triangle on which it stands.

In the color-group III, Fig. 3, the black lines signify a rotation S of period 3 about the diameter passing through the vertex 1 (in Fig. 12), the red lines a rotation T of period 2 about the diameter bisecting the edges 1, 4 and 2, 3.

* Cf. Klein, Icosaëder, page 47.

The *color-group* IV, Fig. 4, shows the generation of the tetrahedron group by two rotations of period 3. The black lines mean the same rotation S of period 3 as in III, the red lines a rotation U of period 3 about the vertex 3.

Comparing Figs. 3 and 4 we recognize immediately the relation $U = TS$, viz. U transforms in Fig. 4, TS in Fig. 3 the point a_1 into d_2 , etc.

The Octahedron (cf. Figs. 5, 6, 7 and 13).

In Fig. 7 the point 3 represents the apex of the upper pyramid of the octahedron, point 4 the opposite vertex. The four triangles concurring in 4 are separated in the figure by sections along the four edges concurring in 4. Every two opposite faces are denoted by equal letters.

The black lines of the *color-group* V, Fig. 5, represent a rotation S of period 4 about the axis 3, 4, the red lines a rotation T of period 2 about the diameter bisecting the edges 3, 5 and 4, 6.

In the *color-group* VI, Fig. 6, the black lines signify a rotation U of period 3 about the diameter passing through the middle points of the two faces a , the red lines a rotation V of period 2 about the diameter bisecting the edges 4, 5 and 3, 6.

Finally, in the *color-group* VII, Fig. 7, the black lines represent the same rotation S of period 4 as in V, and the red lines the same rotation U of period 3 as in VI.

We readily deduce from these three *color-groups* the following relations: $U = S^3 TS^2$, $V = S^2 TS^2$, $S = UV$, $T = VUVU^2 VU$, etc.

The Icosahedron (cf. Figs. 8, 9, 10 and 14).

Fig. 14 represents the icosahedron in two halves. Opposite faces are denoted by equal letters, opposite vertices by numbers whose sum is 12.

In the *color-group* VIII, Fig. 8, the black lines indicate a rotation S of period 5 about the diameter passing through the vertices 0 and 12, the red lines a rotation T of period 2 about the diameter bisecting the edges 0-4 and 12-8.

The black lines in the *color-group* IX, Fig. 9, represent a rotation U of period 3 about the diameter passing through the middle points of the two faces c , the red lines the same rotation T of period 2 as in VIII.

Finally, in the *color-group* X, Fig. 10, the black lines signify the rotation S of period 5 defined in VIII, the red lines the rotation U of period 3 defined in IX.

The relation $U = ST$ is immediately recognized from the diagrams.

The three color-groups of the icosahedron just discussed show that the icosahedron group can be generated by two rotations of periods 2 and 5, 2 and 3, 3 and 5 resp. But the icosahedron group can also be generated by two rotations of period 3. As such we may take for instance a rotation V of period 3 about the diameter passing through the middle points of the two faces a , and a rotation W of period 3 about the diameter passing through the middle points of the two faces b . And now it follows from the preceding developments that the corresponding color-group, consisting of V - and W -lines, can only be such that these V - and W -lines must of necessity intersect each other.

In the simpler case of the octahedron a similar remark takes place. Besides the rotations of V, VI and VII, also the following two rotations, both of period 4, can be taken in order to generate the octahedron group: a rotation V of period 4 about the axis 3, 4 (Fig. 13) and a rotation W of period 4 about the axis 1, 2. The corresponding color-group, in which V is represented by black lines, W by red lines, is given in Fig. 15. It is impossible to disentangle this diagram in such a way that there be no intersection of the color lines.

§6.—*The Groups IIa and VIa.*

The color-groups IIa consist of two black concentric polygons of n sides each, whose opposite vertices are connected by red lines. The black lines represent an operation S of period n , the red lines an operation R of period 2. The two black polygons are directed the same way (in the plane). If Fig. 2 is to represent the case $n = 3$, then the arrows of the outside triangle must be reversed.

It appears at once from the diagram that any two operations of these groups are commutative. If n is odd, then it can readily be seen that the group is a cyclic group of order $2n$, since we have in this case $S = (SR)^{n+1}$, $R = (SR)^n$. If, however, n is even, then we apply the following general theorem* concerning commutative groups:

If all the operations of a group are commutative, there is a system of generating operations S_1, S_2, S_3, \dots which possesses the property that the products

$$S_1^{h_1} S_2^{h_2} S_3^{h_3} \dots \quad (h_i = 1, 2, 3, \dots, r_i)$$

* Kronecker, Berl. Monatsberichte, 1870, p. 881; Netto, Theory of Substitutions, translated by Cole, p. 159.

include every operation of the group once and only once. The numbers r_1, r_2, r_3, \dots are the periods of S_1, S_2, S_3, \dots and are such that every one is divisible by the next following. The product of these periods r_1, r_2, r_3, \dots is equal to the order of the group.

In our case, putting $n = 2m$, we may take

$$S_1 = S, \quad S_2 = R,$$

then there is $r_1 = 2m, r_2 = 2, S^{2m} = 1, R^2 = 1$ and the order of the group $r_1 r_2 = 4m$. The operations of the group are indeed given by

$$1, S, S^2, \dots, S^{2m-1}, R, SR, S^2R, \dots, S^{2m-1}R.$$

For $m = 1$ we obtain the four-group. For higher values of m there exists always a four-group as a (self-conjugate) subgroup, given by the operations

$$1, S^m, R, S^mR,$$

which appears immediately from the diagram.

The color-group VIa, whose diagram is given in Fig. 6, where the arrows are to be changed so that the black triangles have alternately opposite directions, represents a group of order 24 which is not related at all to the octahedron group. This appears from the fact that the group VIa contains no operations of period 4; it contains 8 operations of period 6, 8 operations of period 3, and 7 of period 2. The group is very closely connected with the tetrahedron group, which it contains as a self-conjugate subgroup. This relation will be shown in §7.

PART III.

§7.—*The Extended Rotation Groups.*

The principles underlying the above given results are capable of being extended in several directions. The following remark leads to such an extension:

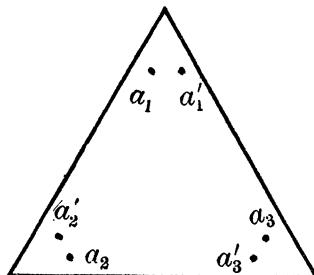
The color-groups consisting of only one generating operation are given by plane one-colored polygons. The simplest method to obtain two-colored groups from these one-colored groups is suggested by the dihedron diagram in Fig. 2: Join every two opposite vertices of two concentric one-colored polygons by a new color line of period 2.

Let us now apply the same principle to the two-colored groups. Take the tetrahedron, octahedron and icosahedron with truncated vertices, polyhedrons which, as we know, represent two-colored groups, surround every one by a similar concentric polyhedron, and join every two opposite vertices by a new, say blue, color line of period 2.

Instead, we may retain the color-groups III, V, VIII in the plane. We lay out in a second sheet, say parallel to the first, the same diagrams and join corresponding points by blue lines.

In this manner we obtain precisely the so-called *extended rotation groups** of the regular bodies, interpreting the blue lines as a reflection on any plane of symmetry and fixing the arrows on the outside sphere, or upper sheet, opposite to those on the inside sphere, or lower sheet.

We have to deal now with a full system of $2N$ points. The faces of the three above-named polyhedrons are triangles. There are two points of the full system in the vicinity of every vertex on every triangle. We keep for the points of the old system of N points the old notation, and distinguish the new points, symmetrically situated with respect to the old ones, by accents. The



adjoined figure shows, for instance, the triangle a of Fig. 12. Each point characterizes one of the $2N$ regions into which one fundamental region† is converted by the $2N$ transformations of the group.

In the case of the tetrahedron we define the blue lines as a reflection on the symmetry plane passing through the edge 1-4 (Fig. 12). The effect will be this permutation of the points: $(a_1 a'_1)$, $(a_2 a'_3)$, $(b_1 c'_1)$, $(b_3 c'_2)$, $(b_4 c'_4)$, $(d_2 d'_3)$, $(d_4 d'_4)$, etc.

Above all we have to ascertain whether the diagram described in the upper sheet with the accented points as fundamental points represents again the origi-

* Klein, Icosaëder, p. 23.

† Klein, Icosaëder, p. 23.

nal tetrahedron group. But this follows at once when we observe that the upper diagram can be obtained by reflecting the original diagram III, Fig. 3, on the symmetry line passing through the points a_1, d_4 , and by attaching accents to the letters.

This three-colored, three-dimensional diagram, as a whole, represents indeed a color-group, as the truth of theorems 1 and 2 (§1) applied to our case can easily be verified.

According to what has been said about the arrows in the upper sheet, every two corresponding triangles, together with the blue lines joining their corners, represent a dihedron color-group ($n = 3$) analogous to the diagram II, Fig. 2.

All these propositions hold equally true in the case of the octahedron, icosahedron and dihedron.

If, in the case of the octahedron, the blue lines indicate a reflection on the symmetry plane passing through the points 5, 3, 6, 4 (Fig. 13), then the upper sheet of the color-group can be obtained by a reflection of the diagram V, Fig. 5, on the symmetry line bisecting the lines $a_4 b_4$ and $c_4 d_4$.

In the case of the icosahedron we reflect on a symmetry plane passing through the edges 0–4 and 12–8 (Fig. 14). Then the upper sheet of the color-group is obtained by the reflection of the diagram VIII, Fig. 8, on the symmetry line passing through the points a_{12}, f_8, a_0, f_4 .

The extension of the dihedron groups I and II hardly requires any further explanation.

We obtain three-colored three-dimensional groups of an entirely different character when we agree to let the arrows in the upper sheet run the *same* way as in the lower sheet. I mention only two cases.

First, the color-group of type IIa, for $n = 2m$, furnishes, when extended in this way, a *commutative group of order $8m$* .

In the second place, the tetrahedron color-group III, Fig. 3, is extended by this method to a three-colored group of order 24, which is holohedrally isomorphic to the color-group VIa explained at the end of §6. To establish this isomorphism, it suffices to show the connection between the generating operations—colors—of the two groups. Denoting in the group VIa the generating operation of period 3, converting a_2 into a_3 , c_2 into b_6 , c_1 into d_4 , b_1 into d_5 (cf. the modified Fig. 6) by S , and by R the operation indicated in the same figure

by red lines, and putting

$$RSRS^{-1} = U, \quad (RS)^3 = B$$

(U and B are both of period 2), then S and U generate a tetrahedron color-group of type III, and taking now B for the blue lines, we obtain precisely the three-dimensional three-colored group of order 24 described above. We observe that all the triangles of the diagram VIa of one direction lie in one sheet of the three-colored group, all those of the other direction in the other sheet.

§8.—Three-colored “Regular” Color-groups.

The color-group of the extended dihedron, obtained by the methods of §7 from diagram I, can be spread out in a plane, as shown in Fig. 16, where the blue lines are represented as dotted lines, The group affords an example of a “regular” three-colored group, i. e. a color-group of three colors whose lines do not meet except in the fundamental points.

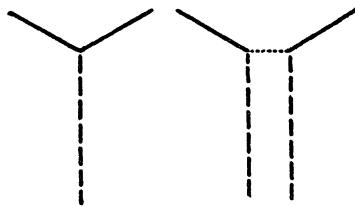
With regard to these regular three-colored groups there holds the general theorem, which is readily seen to be true in the special case represented by Fig. 16, that *every one of the three generators of such a group must be of period 2*.

To show this, let us denote the three colors by S (black), R (red) and B (blue). If now we drop one of the colors, say the B -lines, there remain two-colored groups C_2 , i. e. diagrams given by one of the Figs. 1–10.

To fix the ideas, let the groups C_2 be tetrahedron color-groups of the type III, Fig. 3. Consider now the B -lines emanating from the corners of the triangle in the center of one of the diagrams III. These lines cannot lead to any other point of III, otherwise B would be expressible in terms of S and R . The B -lines must, therefore, terminate somewhere in the interior of III. Again, the end points of the B -lines must hang together with other tetrahedron groups C_2 , which must lie entirely inside the diagram III just considered, without having any point with III in common. But this conclusion can be repeated with regard to every one of these interior tetrahedron groups C_2 , etc., *in infinitum*.

Consequently the color-groups C_2 cannot be tetrahedron groups. But the same demonstration applies to octahedron, icosahedron and dihedron groups of type II (including evidently also the cases VIa and IIa), and that means, there cannot exist any color lines of periods higher than two. Q. E. D.

Besides the extended dihedron groups, Fig. 16, we obtain other regular three-colored groups in the following way:



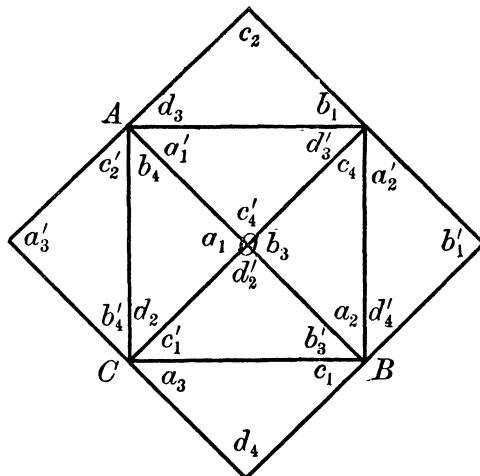
In the diagrams III, V, VI, VIII and IX, given by Figs. 3, 5, 6, 8, 9, we truncate the corners of the black polygons by blue lines and replace accordingly every red line by two red lines, as shown in the adjoined figures, where the left figure shows the original, the right one the modified form.

This process amounts to the same effect as that produced by truncating on the regular bodies first the edges and then the vertices so far as to obtain polygons of twice as many sides as there were edges concurring in the original vertices.

The resulting diagrams are three-colored regular color-groups, and these color-groups represent again the extended rotation groups of the regular bodies, as will appear from the examination of the single cases.

Fig. 17 shows the *extended tetrahedron group* with the notation given in §7. The blue lines signify a reflection B on the symmetry plane passing through the edge 1-4 (Fig. 12), the red lines the operation BT , where T represents the rotation of period 2, defined by red lines in diagram III (cf. Fig. 3 and page 172), the black lines the operation BS , where S represents the rotation of period 3, defined by black lines in III (cf. Fig. 3 and page 172).

The same color-group (Fig. 17) represents also the octahedron group.



This remarkable relation appears indeed when we denote the 24 fundamental points of the octahedron group as shown in the adjoined figure, and when now we interpret the blue lines (Fig. 17) as a rotation of period 2 about the middle of OA , the red lines as a rotation of period 2 about the middle of OB , and the black lines as a rotation of period 2 about the middle of BC .

*The extended tetrahedron group is therefore holohedrally isomorphic to the octahedron group.**

The three-colored regular color-group belonging to *the extended octahedron group* is represented in Fig. 18 with the notations given in §7. It may be derived as well from the diagram V, Fig. 5, as from VI, Fig. 6, by the above defined process. The blue lines indicate the rotation T of diagram V (cf. page 173), the black lines the rotation STS^2T , where S of period 3 is defined by the black lines in V (cf. page 173), and the red lines a reflection on the symmetry plane passing through the vertices 4, 6, 3, 5 (Fig. 13).

Finally, the extended icosahedron group can be derived by our process from diagram VIII, Fig. 8, or IX, Fig. 9, using the following three operations of period 2:

- 1) a reflection B on any symmetry plane;
- 2) the operation BS , where S is defined by the black lines of VIII (cf. page 173);
- 3) the operation BT , where T is defined by the black lines of VIII (cf. page 173).

While the extended tetrahedron group, as we have seen, is holohedrally isomorphic with the octahedron group, i. e. the symmetric group of 4 elements, such an isomorphism does not exist between the extended icosahedron group and the symmetric group of 5 elements.

PART IV.

§9.—*The Rotation Groups of the Regular Bodies of Four-dimensional Space.*

We obtained the color diagrams of the rotation groups of the regular three-dimensional bodies by truncating the vertices or edges of these polyhedrons. The analogous process applied to the regular four-dimensional bodies leads to a

* I have not found this fact stated anywhere else.

color representation of the rotation groups converting these bodies into themselves.

First of all, we project the four-dimensional bodies into three-dimensional space, so that we obtain the well-known Schlegel-Brill models, which, on their part, define the four-dimensional bodies* uniquely, viz. :

- the 5-cell*, consisting of 5 tetrahedrons, 5 vertices, 10 faces and 10 edges ;
- the 8-cell*, with 8 hexahedrons, 16 vertices, 24 faces and 32 edges ;
- the 16-cell*, with 16 tetrahedrons, 8 vertices, 32 faces and 24 edges ;
- the 24-cell*, with 24 octahedrons, 24 vertices, 96 faces and 96 edges ;
- the 120-cell*, with 120 dodecahedrons, 600 vertices, 720 faces and 1200 edges ;
- the 600-cell*, with 600 tetrahedrons, 120 vertices, 1200 faces and 720 edges.

The 16-cell, dual to the 8-cell, and the 600-cell, dual to the 120-cell, yield the same rotation groups as their duals, and may accordingly be omitted in the following.

We now apply the process of truncation to the three-dimensional projections of the regular bodies, so that we obtain color diagrams situated in three-dimensional space, in perfect analogy with the two-dimensional diagrams (Figs. 1-10) representing the truncated three-dimensional regular bodies.

In every vertex of the 5-cell, 8-cell and 120-cell there are 4 concurring edges ; in every vertex of the 24-cell, 8 edges. To fix the ideas, we assume these 4 bodies constructed of blue threads, and now we replace every vertex of the 5-, 8- and 120-cell by tetrahedrons, every vertex of the 24-cell by hexahedrons—which polyhedrons we think of as constructed of red threads—in such a manner that, for instance, the 4 blue threads, originally concurring in one vertex of the 120-cell, are now attached to the vertices of the substituted tetrahedron. In this way we obtain, reverting to four-dimensional space, four-dimensional bodies whose vertices are, so to speak, truncated by three-dimensional spaces.

But while the analogous process performed on the three-dimensional regular bodies, explained in §5, was already sufficient in order to furnish color-groups, since in that case the polygons substituted in place of the vertices represented already one-colored color-groups, we have now to proceed one step further. In

* Cf. Schlegel, Theorie der homogenzusammengesetzten Raumgebilde, Nova Acta d. Kais. Leop.-Carol. deutschen Academie, Bd. 44, No. 4 ; Stringham, On Regular Figures in n -Dimensional Space, American Journal, vol. II ; Van Oss, Die Bewegungsgruppen der regelmässigen Gebilde von vier Dimensionen, Dissertation, Giessen, 1894 ; a. o.

the substituted tetrahedrons and hexahedrons we again truncate their vertices, i. e. we replace every (red) vertex by a triangle consisting of black threads. Finally we have to replace every one of the blue threads leading originally to the vertices of the red polyhedrons by three, say parallel, blue threads leading now to the three corners of the black triangles which have been substituted in place of the (red) vertices.

The three-colored thread models thus obtained represent indeed color-groups, if we define the direction of the lines as follows: The direction of the black lines is such that every black triangle is circumscribed by the arrows *positively* when the truncated tetrahedron or hexahedron on which the triangle lies is viewed from an outside point. The blue and red lines are both double lines, i. e. of period 2. We now have leading *from* and *to* every point of our model a blue, a red, and a black line. Again it is a consequence of the regularity of the corresponding four-dimensional body that every route leading back from one point to the same point will lead back from every point to the same point.

We have now before us 4 three-colored groups, two of whose generating operations are of period 2, one of period 3. The number of the resp. fundamental points, i. e. the points in which the black, red, and blue lines meet, gives us the order of the group concerned, viz. 60 for the 5-cell, 192 for the 8-cell, 576 for the 24-cell, and 7200 for the 120-cell, because these points constitute general point systems on the regular bodies.

These color-groups are holohedrally isomorphic to the rotation groups converting the regular four-dimensional bodies into themselves.

There is a certain degree of choice as to the definition of rotations in four-dimensional space.* If we fix the vertices of the regular bodies by the 4 coordinates x, y, z, w of a coordinate system whose origin lies in the center of the resp. regular body, then these vertices lie on a spherical space

$$x^2 + y^2 + z^2 + w^2 = \text{const.}$$

Let us agree then to define a rotation about the center by an orthogonal substitution of the 4 coordinates x, y, z, w with +1 as determinant of substitution.

But these orthogonal, quaternary substitutions can also be interpreted as collineations of three-dimensional space with regard to the homogeneous vari-

* Cf. Cole, On Rotations in Space of Four Dimensions, American Journal, vol. XII, p. 191.

ables x, y, z, w , leaving the sphere

$$x^2 + y^2 + z^2 + w^2 = 0 \quad (19)$$

unaltered.

Now it is a known fact that every collineation of ordinary space that converts a quadric surface into itself can be reduced in a certain manner to a combination of two binary substitutions.* To show this, let us consider the two series of straight lines on the quadric surface S_2 , which may be defined by the parameters ξ and η . Every collineation converting the S_2 into itself transforms either the lines of the ξ -system and those of the η -system into themselves, or it interchanges the two systems. Accordingly the parameter ξ is transformed by a linear, non-homogeneous substitution, and also the parameter η or ξ is transformed linearly into η , and vice versa.

In the case where the quaternary substitutions defining the collineation in question are *orthogonal*, i. e. where the S_2 is given by equation (19), the connection between the linear substitutions of the quantities ξ, η , which we may write homogeneously as $\xi = \xi_1 : \xi_2$, $\eta = \eta_1 : \eta_2$, and those of the quantities x, y, z, w , can be brought about by the following formulæ:

$$\begin{aligned} \rho x &= \xi_1 \eta_2 + \xi_2 \eta_1 \quad \text{and} \quad \sigma \cdot \xi_1 \eta_1 = y + iz, \\ \rho y &= -\xi_1 \eta_1 + \xi_2 \eta_2 \quad \sigma \cdot \xi_1 \eta_2 = -x - iw, \\ \rho z &= i(\xi_1 \eta_1 + \xi_2 \eta_2) \quad \sigma \cdot \xi_2 \eta_1 = -x + iw, \\ \rho w &= i(-\xi_1 \eta_2 + \xi_2 \eta_1) \quad \sigma \cdot \xi_2 \eta_2 = -y + iz, \end{aligned} \quad (20)$$

where ρ and σ are factors of proportionality.

Again it can be shown that if the determinant Δ of the orthogonal substitution is $= +1$, then the first of the two above-mentioned cases holds, viz. ξ is transformed linearly, and also η , while in the case $\Delta = -1$, ξ is linearly transformed into η and η into ξ .

There corresponds consequently to every group of rotations converting one of the four-dimensional regular bodies into itself, an isomorphic group of simultaneous binary substitutions of the form

$$\xi' = \frac{a\xi + b}{c\xi + d}, \quad \eta' = \frac{b\eta + m}{n\eta + p}. \quad (21)$$

* See Goursat, *Sur les substitutions orthogonales et les divisions régulières de l'espace*, Ann. de l'École Norm. Sup., 1889; cf. also Clebsch-Lindemann, *Vorlesungen über Geometrie*, Bd. II, p. 256 ff.

The isomorphism is hemihedral as soon as, in the quaternary substitution group representing the rotation group, a simultaneous change of signs of all the 4 variables occurs. Such a change of signs has no effect upon the formulæ (20), so that indeed to every substitution (21) there correspond two substitutions of the quantities x, y, z, w .

Since the groups in consideration are all of finite orders, the groups formed by the binary substitutions (21), taken with respect to the single quantities ξ or η , must be contained among the known rotation groups of the regular three-dimensional bodies. The ξ 's and the η 's may be substituted according to the same or to different groups. A complete solution of the problem to find all the finite groups possible, formed by the simultaneous binary substitutions of two quantities ξ and η , has been given by Goursat, l. c.

In order to establish the isomorphism between the 4 above defined color-groups and the four-dimensional rotation groups on one hand and the resp. ξ, η -groups on the other, I shall in the following define analytically the 3 generating operations S of period 3, represented by the black lines, R of period 2, represented by the red lines, and B of period 2, represented by the blue lines. With regard to the values of the coordinates by which the vertices of the regular four-dimensional bodies are determined, we avail ourselves of a paper by Puchta,* where a complete table of these values can be found.

The 5-cell.

Coordinates of the vertices 1–5:

	1	2	3	4	5
x	1	1	–1	–1	0
y	1	–1	–1	1	0
z	1	–1	1	–1	0
w	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	$\frac{-4}{\sqrt{5}}$

* Puchta, Analytische Bestimmung der reg. conv. Körper im Raum von vier Dimensionen, Wiener Sitzungsberichte, Bd. 89, 1884.

For S, R, B we take in the quaternary group these substitutions:

$$S \begin{cases} x' = y, \\ y' = z, \\ z' = x, \\ w' = w, \end{cases} \quad R \begin{cases} x' = -x, \\ y' = y, \\ z' = -z, \\ w' = w, \end{cases} \quad B \begin{cases} -4x' = x + y - 3z + \sqrt{5} \cdot w, \\ -4y' = x - 3y + z + \sqrt{5} \cdot w, \\ -4z' = -3x + y + z + \sqrt{5} \cdot w, \\ -4w' = \sqrt{5} \cdot x + \sqrt{5} \cdot y + \sqrt{5} \cdot z + \sqrt{5} \cdot w. \end{cases}$$

S and R generate indeed a tetrahedron group consisting of those 12 substitutions which are obtained by combining all the even permutations of x, y, z with an even number of changes of signs. The vertices 1-5 of the 5-cell are permuted by S, R, B as follows:

$$S \equiv (234), \quad R \equiv (14)(23), \quad B \equiv (15)(23).$$

Hence the rotation group of the 5-cell is precisely the icosahedron group consisting of the even permutations of the vertices 1-5.

According to the color-group, SB must be of period 2 and RB of period 3; we find indeed $SB \equiv (15)(34)$ and $RB \equiv (145)$. S and B generate a dihedron group ($n = 3$) whose color-group consists of the black and blue lines.

The ξ, η -substitutions are given by these formulæ:

$$S \begin{cases} \xi' = i \frac{\xi + 1}{\xi - 1}, \\ \eta' = i \frac{\eta + 1}{\eta - 1}, \end{cases} \quad R \begin{cases} \xi' = \frac{1}{\xi}, \\ \eta' = \frac{1}{\eta}, \end{cases} \quad B \begin{cases} \xi' = \frac{2\xi - (1 - \sqrt{5}) \left[-1 + \frac{i}{2} (1 - \sqrt{5}) \right]}{(1 - \sqrt{5}) \left[1 + \frac{i}{2} (1 - \sqrt{5}) \right] \xi - 2}, \\ \eta' = \frac{2\eta - (1 + \sqrt{5}) \left[-1 + \frac{i}{2} (1 + \sqrt{5}) \right]}{(1 + \sqrt{5}) \left[1 + \frac{i}{2} (1 + \sqrt{5}) \right] \eta - 2}, \end{cases}$$

ξ and η alone are substituted according to an icosahedron group. Every ξ -substitution is combined with that η -substitution which is obtained from the ξ -substitution by interchanging $+\sqrt{5}$ with $-\sqrt{5}$. The group is identical with Goursat's group No. XXXII.*

The ξ, η -group is *holohedrally* isomorphic to the rotation group, because in the corresponding quaternary substitution group a simultaneous change of signs does not occur.

*l. c., page 68.

The 8-cell.

The coordinates of the 16 vertices have the values

$$x, y, z, w = \pm 1, \pm 1, \pm 1, \pm 1.$$

We take here

$$S \begin{cases} x' = y, \\ y' = w, \\ z' = z, \\ w' = x, \end{cases} \quad R \begin{cases} x' = y, \\ y' = x, \\ z' = w, \\ w' = z, \end{cases} \quad B \begin{cases} x' = y, \\ y' = x, \\ z' = -z, \\ w' = w. \end{cases}$$

S and *R* generate a tetrahedron group consisting of all the even permutations of x, y, z, w . Combining the same with *B*, we obtain the complete group. *The rotation group of the 8-cell consists therefore of all those substitutions that are obtained by forming the symmetric permutation group of the 4 quantities x, y, z, w , combined with all changes of signs possible and retaining those of determinant +1.*

The order is accordingly, as it ought to be, $\frac{24 \cdot 24}{2} = 192$. As the formulæ

show, *SB* is of period 2, *RB* of period 4, which agrees with the color-group. *S* and *B* generate a dihedron ($n = 3$), etc.

The ξ, η -group is given by

$$S \begin{cases} \xi' = \frac{i - \xi}{i + \xi}, \\ \eta' = \frac{i - \eta}{i + \eta}, \end{cases} \quad R \begin{cases} \xi' = -\xi, \\ \eta' = \frac{1}{\eta}, \end{cases} \quad B \begin{cases} \xi' = \frac{\xi + 1}{\xi - 1}, \\ \eta' = \frac{\eta + 1}{\eta - 1}. \end{cases}$$

This is Goursat's Group No. XXVII* of order 96; it is hemihedrally isomorphic to the rotation group, since the latter contains indeed a simultaneous change of signs of x, y, z, w .

The 24-cell.

The 24 vertices of the 24-cell have the coordinates

$$x, y, z, w = \pm 1, \pm 1, 0, 0.$$

*l. c., page 67.

We take

$$S \begin{cases} x' = y, \\ y' = -z, \\ z' = -x, \\ w' = w, \end{cases} \quad R \begin{cases} x' = y, \\ y' = x, \\ z' = -z, \\ w' = w, \end{cases} \quad B \begin{cases} 2x' = -x + y + z + w, \\ 2y' = x - y + z + w, \\ 2z' = x + y + z - w, \\ 2w' = x + y - z + w. \end{cases}$$

S and R generate the octahedron group consisting of all the substitutions of determinant $+1$ that are contained in the group which arises by combining all permutations of x, y, z with all possible changes of signs of these 3 quantities. BS is of period 2, BT of period 3.

The ξ, η -group is given by

$$S \begin{cases} \xi' = i \frac{1 + \xi}{1 - \xi}, \\ \eta' = i \frac{1 + \eta}{1 - \eta}, \end{cases} \quad R \begin{cases} \xi' = \frac{\xi + 1}{\xi - 1}, \\ \eta' = \frac{\eta + 1}{\eta - 1}, \end{cases} \quad B \begin{cases} \xi' = \frac{i}{\xi}, \\ \eta' = i \frac{\eta + i}{\eta - i}. \end{cases}$$

This group can be described as follows: Substitute the ξ 's and η 's separately according to the tetrahedron group in that normal form which is given in Klein's "Icosaëder," page 42, equations (30a). Now combine every ξ - with every η -substitution. Thus a group of 12.12 substitutions is obtained. If, finally, we associate the substitution $\xi' = i\xi$, $\eta' = i\eta$, we obtain the complete ξ, η -group of order $2 \cdot 12 \cdot 12 = 288$, Goursat No. XXVIII,* which is hemihedrally isomorphic to the rotation group of the 24-cell of order 576.

The following formulæ show that the ξ, η -substitutions, generated by S, R, B , form indeed the group of order 288 just described. We find

$$(BRS^2)^2: \begin{cases} \xi' = \xi, \\ \eta' = \frac{\eta + i}{\eta - i}, \end{cases} \quad (SBR)^3: \begin{cases} \xi' = \xi, \\ \eta' = -\frac{1}{\eta}, \end{cases}$$

$$(SBR)^2: \begin{cases} \xi' = i \frac{1 - \xi}{1 + \xi}, \\ \eta' = \eta, \end{cases} \quad (BRS^2)^3: \begin{cases} \xi' = \frac{1}{\xi}, \\ \eta' = \eta, \end{cases}$$

$$(RS)^2: \begin{cases} \xi' = i\xi, \\ \eta' = i\eta. \end{cases}$$

* 1. c., page 68.

The two substitutions $(BRS^2)^2$ and $(SBR)^3$ leave ξ unchanged and generate with regard to η the tetrahedron group, while $(SBR)^2$ and $(BRS^2)^3$ leave η unchanged and generate with regard to ξ the same tetrahedron group.

The 120-cell.

The rotation group of the 120-cell, which is isomorphic to the color-group of order 7200 generated by S, R, B , could be defined in a similar way, as has been done in the preceding cases, by 3 quaternary substitutions S, R, B , and likewise the ξ, η -group of order 3600, hemihedrally isomorphic to it. The computations, however, required for that purpose are rather complicated. With regard to the ξ, η -group the result can easily be anticipated. According to Goursat's investigations, there exists only one ξ, η -group of order 3600 (none of order 7200). We obtain this group—enumerated as No. XXX*—by subjecting ξ as well as η to all the substitutions of an icosahedron group and combining every ξ - with every η -substitution.

UNIVERSITY OF CHICAGO, November 29, 1895.

* I. c., page 68.

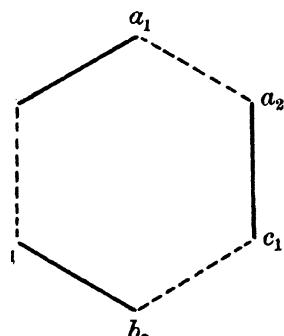


FIG. 1.

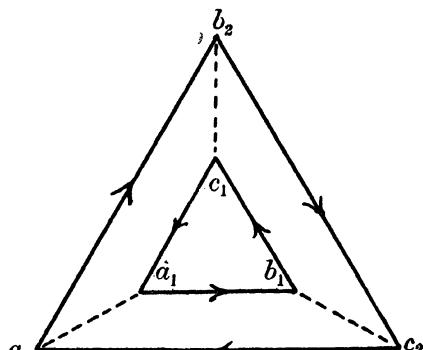


FIG. 2.

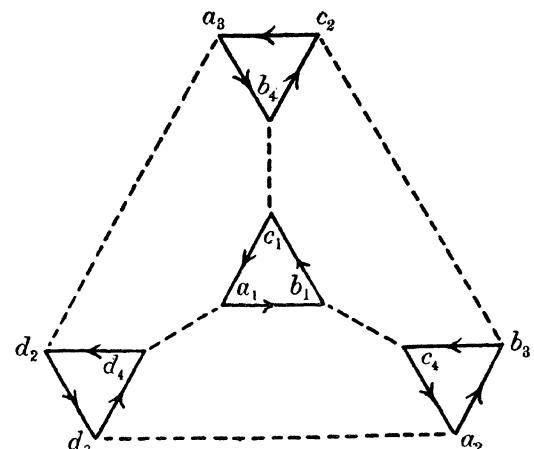


FIG. 3.

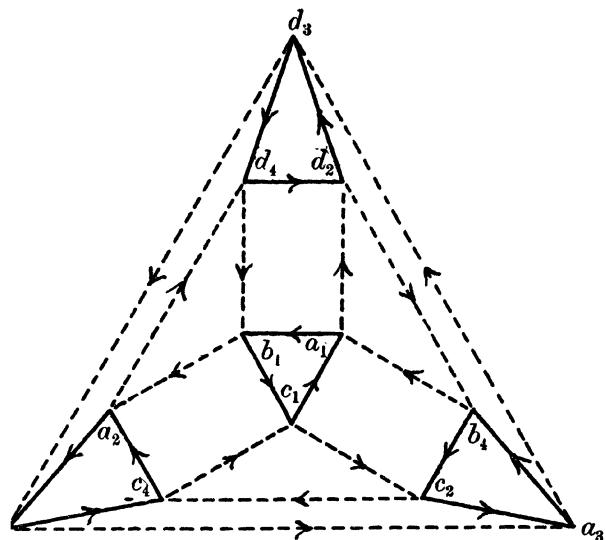


FIG. 4.

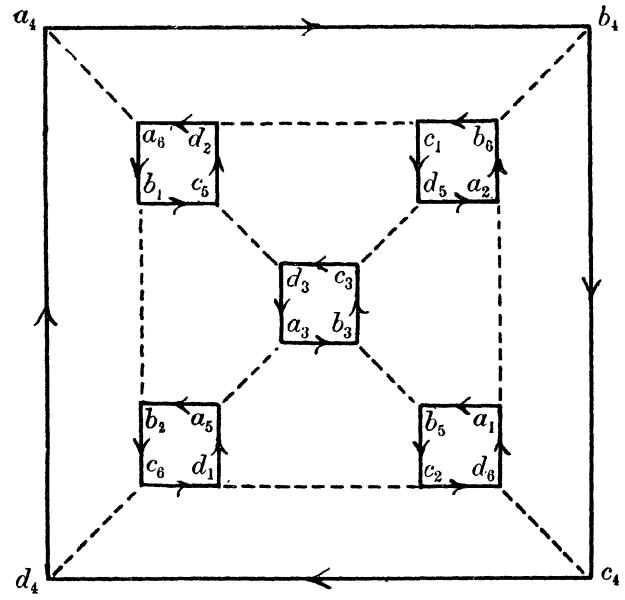


FIG. 5.

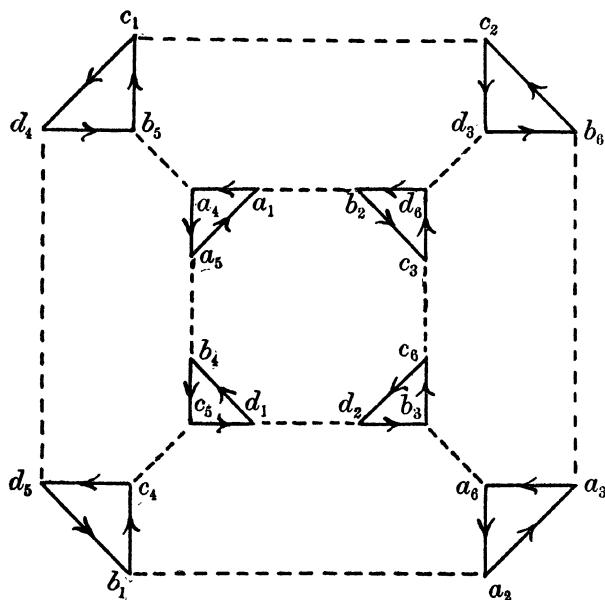


FIG. 6.

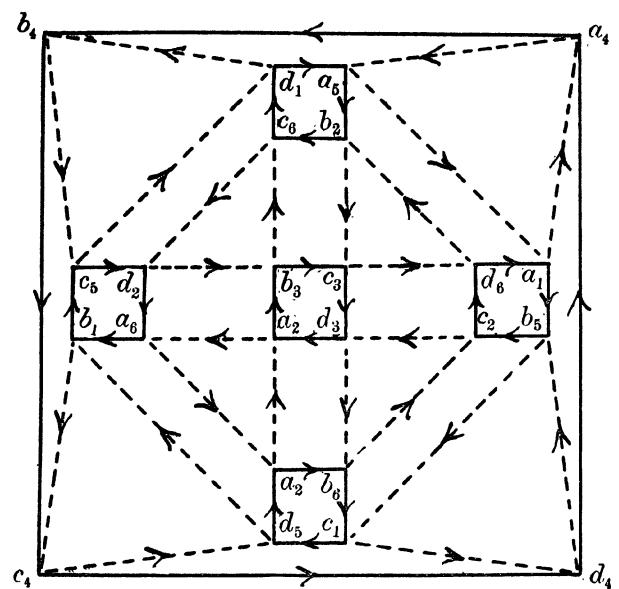


FIG. 7.

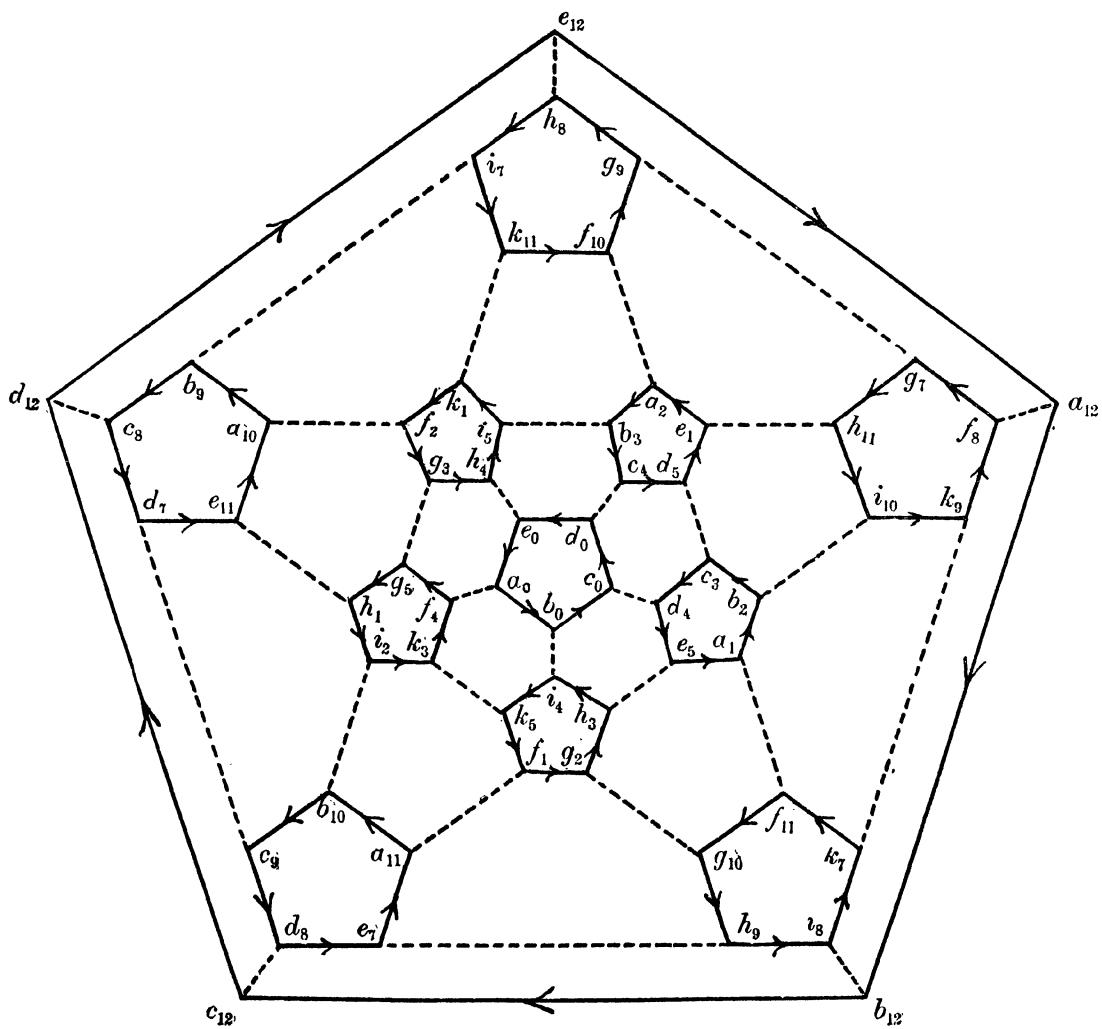


FIG. 8.

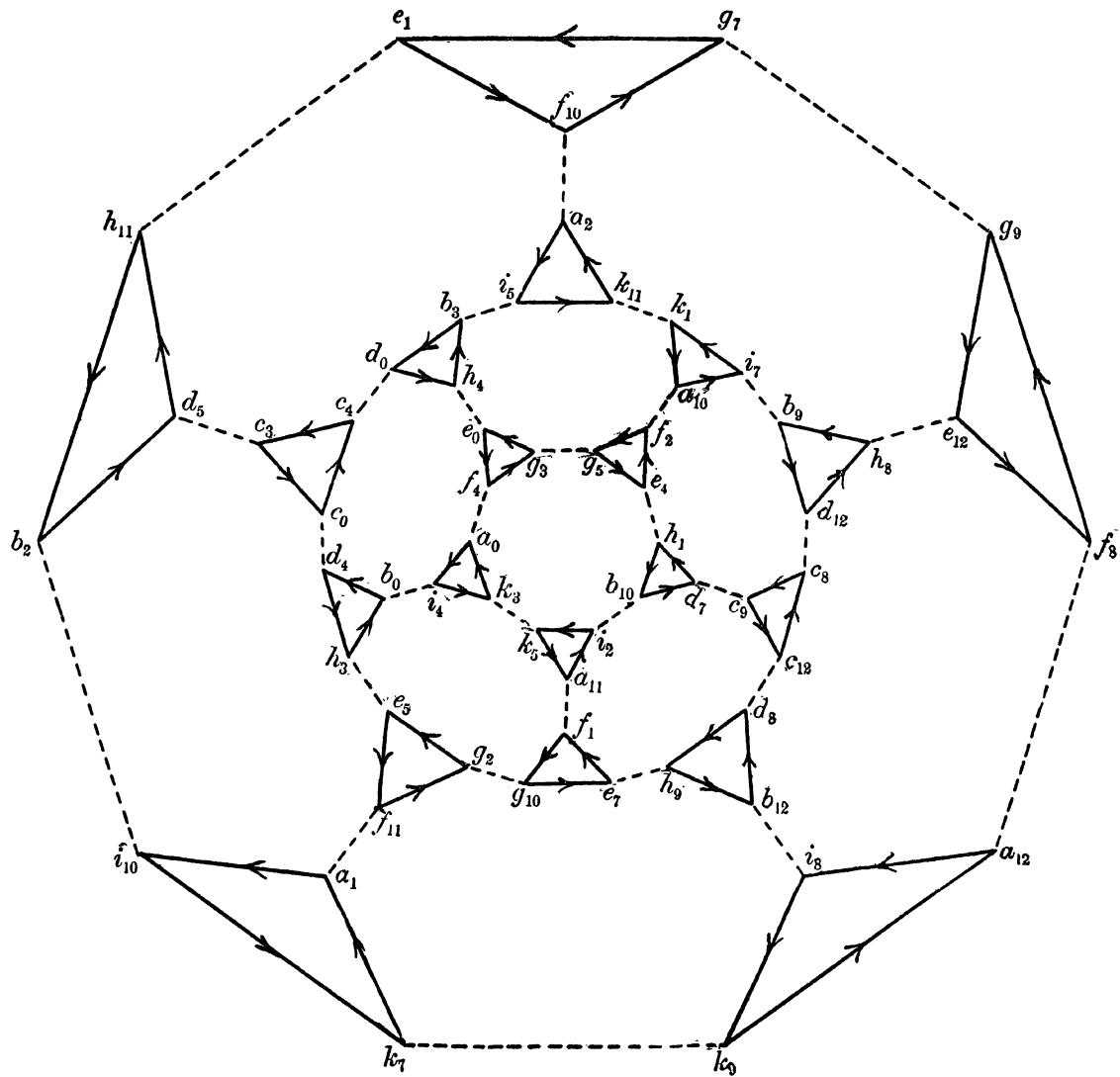


FIG. 9.

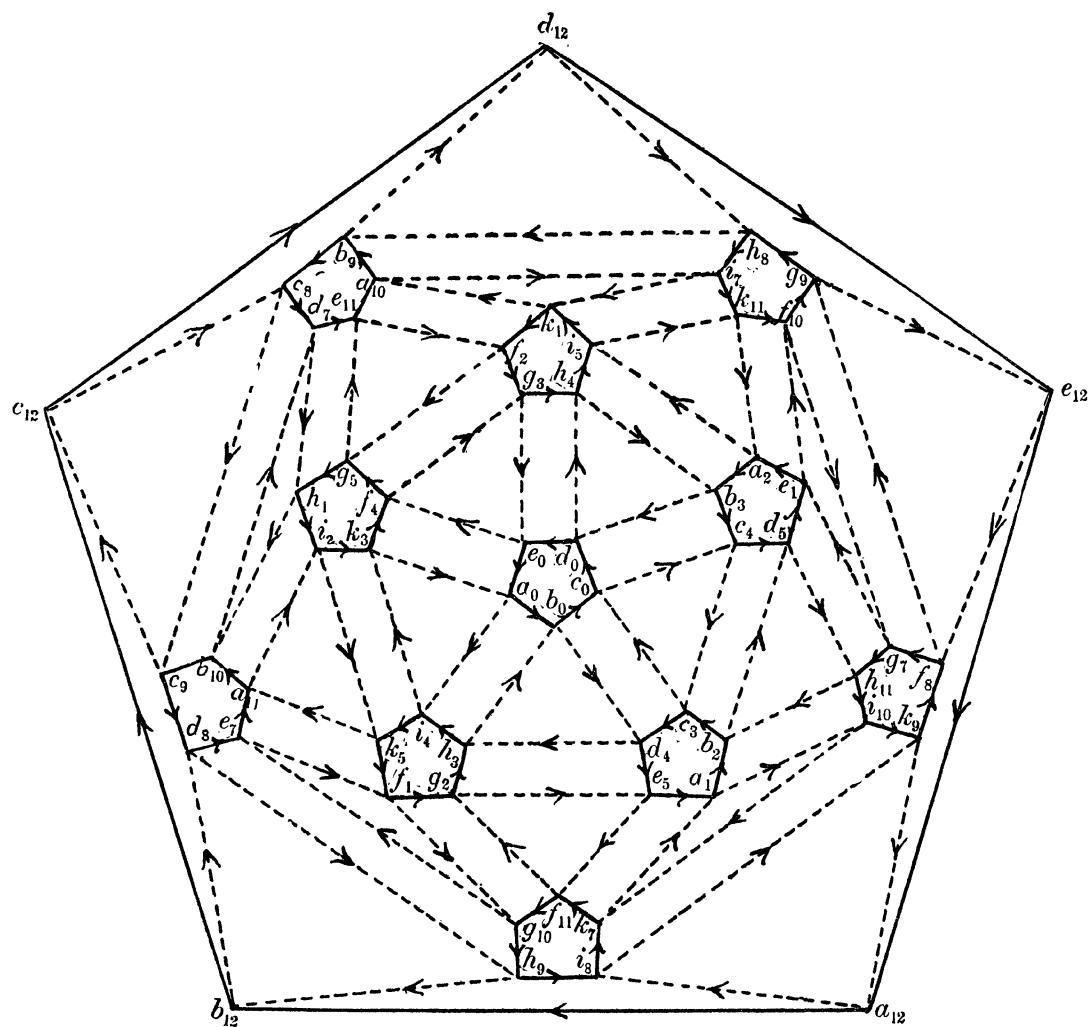


FIG. 10.

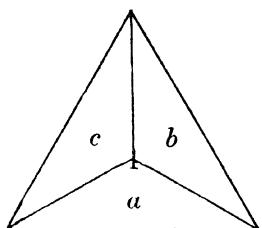


FIG. 11.

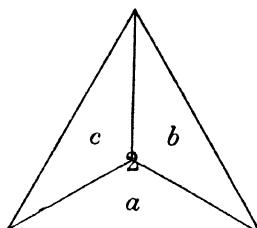


FIG. 11.

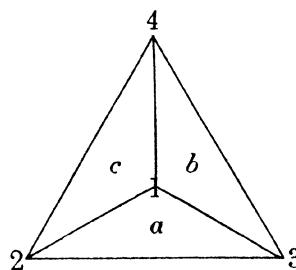


FIG. 12.

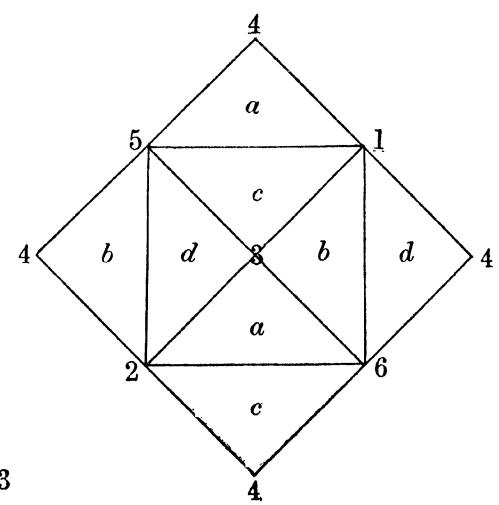


FIG. 13.

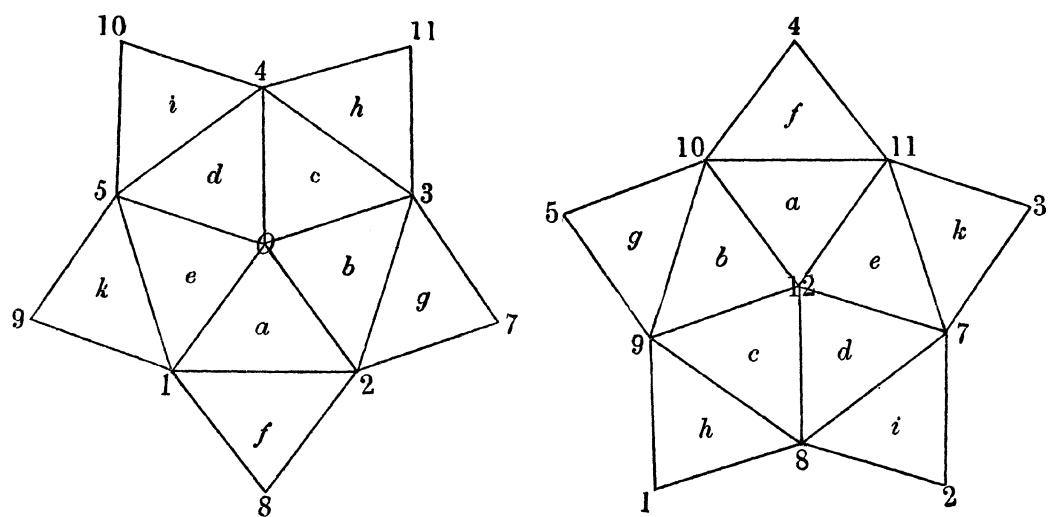


FIG. 14.

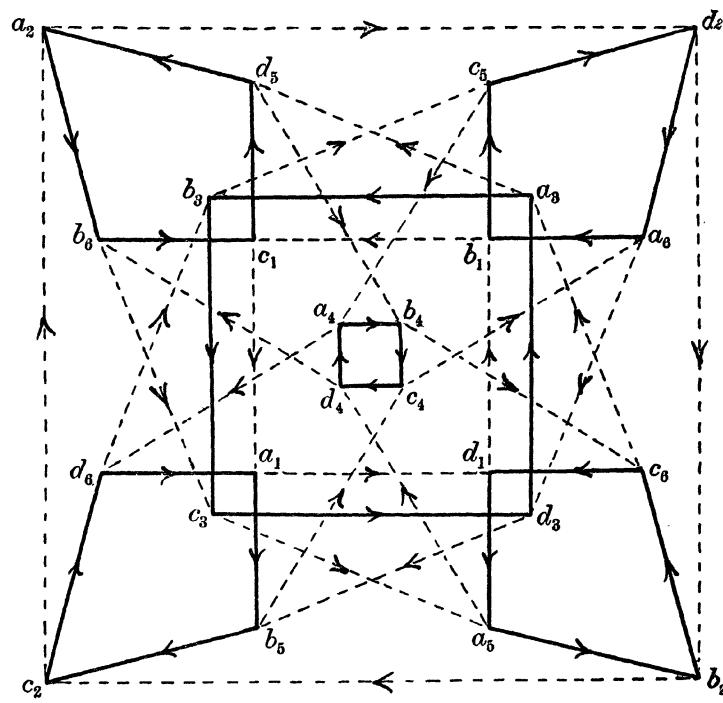


FIG. 15.

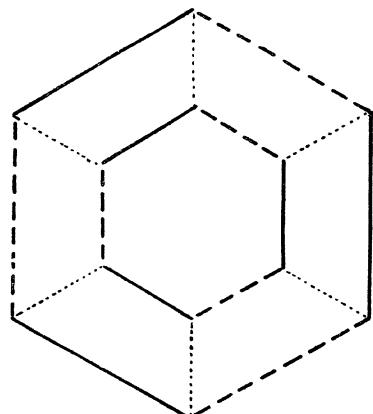


FIG. 16.

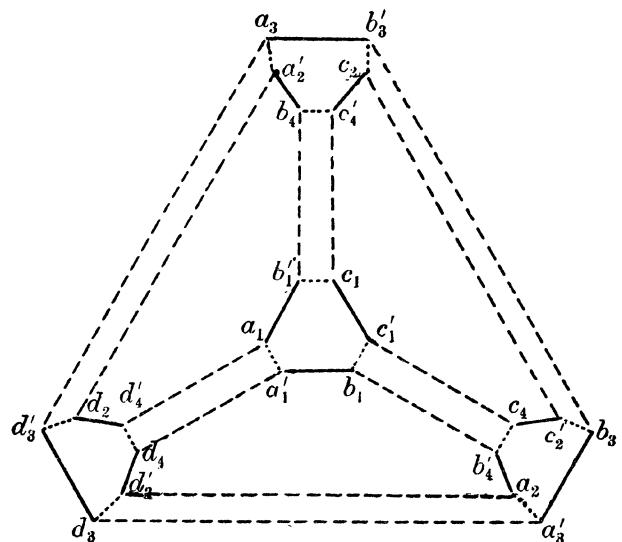


FIG. 17.

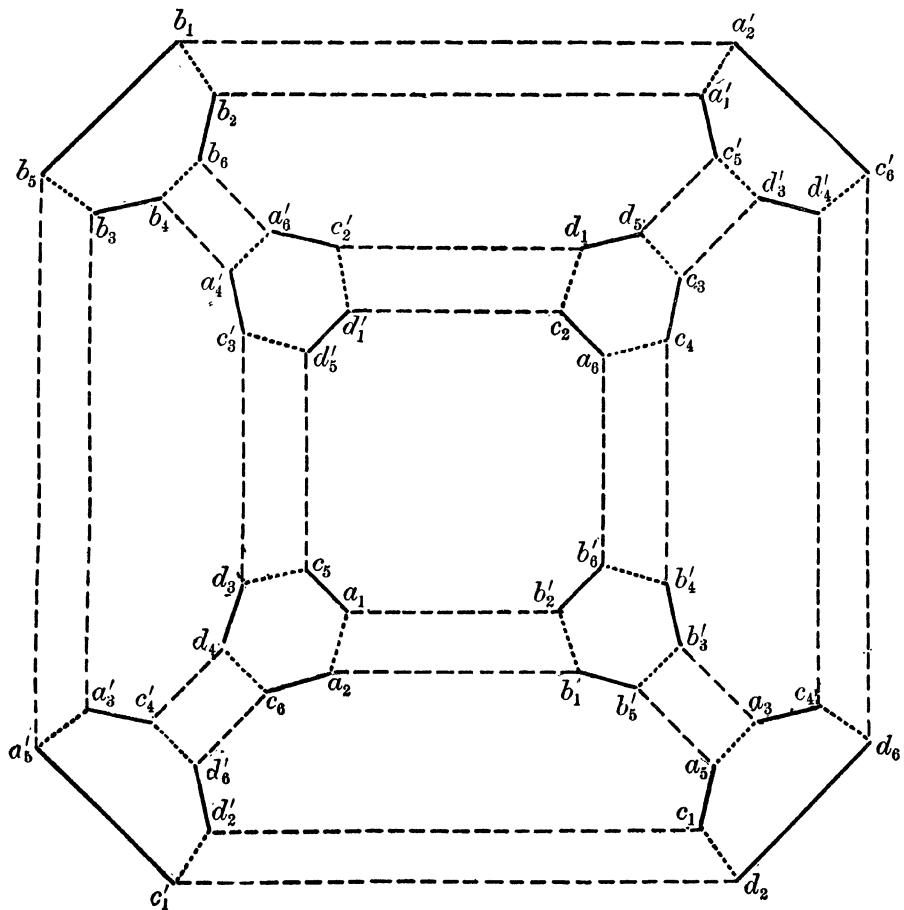


FIG. 18.